YOUNG CHILDREN'S REPRESENTATIONS AND STRATEGIES FOR ADDITION.

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This is a report of a study of the representations and strategies for addition, used by a sample of 55 children in years 1,2, and 3 in three schools in Brisbane. Children were presented with operations represented in symbols and asked to explain their procedures as they worked. The teachers were also interviewed to determine the representations that they were introducing. The general developmental sequence was from use of objects, to use of counting to mental calculations using knowledge of number facts and place value. The results are discussed from the perspective of the demand that the procedures make on children's information processing capacity. We suggest that some of the difficulties occur because teachers introduce procedures that are recommended in curriculum documents without being aware of the cognitive load that they impose.

The intention of the research was to find out how children interpreted the symbols and the addition operation, and how they represented them with analogs and used strategies to solve the problems. The results were analysed from an information processing perspective and compared with the strategies and representations that the teachers introduced. The hypothesis was that some of the strategies and representations that teachers currently accept as good practice do not seem to be as successful as they should be and therefore need closer examination.

THEORIES OF COGNITIVE DEVELOPMENT AND REPRESENTATIONS

Addition is a binary operation which requires three elements and a relation between them to be mapped from one structure to another. It is therefore a concept at the system mapping level and should be possible for children to cognize in its most basic form from about 5 years onwards (Halford, 1988; in press; forthcoming). However not all children have developed the requisite capacity by that age to cognize such a concept and those who have may not perform to the limit of their capacity because of lack of requisite knowledge based on experience. Case and Sowder (1990) predicted, on the basis of the number of mental number lines that the child would need to take into account, that at about 6 years children should become capable of computing single digit sums, at about 8 years they should become capable of mentally computing double digit sums, and at about 10 years they should become capable of computing double digit sums with some form of carrying or regrouping. Their research showed this to be the case. We would explain the complexity of their tasks differently, and not tie the capacities so closely to ages and stages, but rather predict that after about 5 years of age the child should become capable of cognizing binary operations and that subsequent performance would depend on the complexity of the operation in terms of the number of relations that need to be considered and the child's prior knowledge.

Teachers of young children use concrete materials as representations in teaching mathematics in the belief that they will facilitate the learning process. The concrete representation should mirror the structure of the concept and the child should be able to use the structure of the representation to construct a mental model of the concept. In theory concrete materials as analogs should reduce learning effort and serve as memory aids; provide a means of verifying the truth; indirectly facilitate transition to higher levels of abstraction and be used generatively to predict unknown facts. There are however some possible disadvantages. Mapping from a concrete representation to a concept imposes a processing load and this can interfere with the understanding of a concept;

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if an analog is poor or not properly understood it can generate incorrect information; and if it is not well mapped into the material to be remembered it can actually increase the processing or memory load of a task (Halford and Boulton-Lewis, 1992).

There is a growing body of evidence which shows that concrete materials often fail to produce the expected positive outcomes (Boulton-Lewis, in press; Bobis, 1993; Hart, 1989; Sowell, 1989; Lesh, Behr & Post, 1987; Dufour-Janvier, Bednarz & Belanger, 1987; Lesh, Landau & Hamilton, 1983).

Most of the concepts taught in the early years of school, such as addition and subtraction operations require two way mappings if analogs are used. The child is often expected to start with symbols or numbers expressed in word problems, to map the symbols into concrete representations, to perform the operation with the concrete materials and to map the resulting quantity back into symbols (verbal or written). We have described the structure mapping required to show how 324 minus 179 can be understood, as opposed to performed, (Halford & Boulton-Lewis, 1992). In performing that task, to understand how the concrete analog supports the subtraction algorithm the child must cognize a complex set of relationships of relational and system level mappings. The operations require more information to be processed at once than even most adults are capable of. When the complexity of the mappings for three digit operations is analyzed in this way it becomes clear why the loads required to make the mappings in multidigit addition and subtraction could be impossibly high for a young child who did not know all the lower level mappings. The same is true for 2 and 3 digit addition operations.

Sweller and colleagues have undertaken extensive research which has lead them to propose a cognitive load theory which predicts that tasks will be more difficult if there is redundancy in the information which must be processed or if attention must be split between two different sources of information (Tarmizi & Sweller, 1988; Ward & Sweller, 1990). On the basis of his theory and our results for subtraction (Boulton-Lewis, in press) it seems that perhaps the most difficult way to perform two or three digit addition, in terms of the load on processing capacity, is to use analogs, to support limited knowledge of place value, whilst trying to learn to use an algorithm. It is probable that if a child has a good understanding of basic addition as an operation, and understands place value, then mental strategies would make less of a processing demand. For example if a child were to see the problem 265+126 represented in written form, either horizontally or vertically, and reason mentally as follows she would need to hold and process less information at any one time than she would if she needed to make mappings from the concrete representations to the symbols in a written algorithm; 6 and 5 are 1 more than 10 so there's one unit, 9 tens and 3 hundreds, that is 391.

The explanation that is proposed then for the difficulties often experienced with concrete representations, and children's apparent reluctance to use them, is that previous analyses have taken insufficient account of the processing loads, over and above the processing load of the basic operation, that their use initially entails. This implies that children often will not derive the anticipated benefits of concrete representations because, if the analogs are not well known, then they make an extra demand on processing capacity and the processing load of the mapping can become too high for the cognitive resources they can apply to the task. In order to reduce the load it is important to ensure that the child understands the operation, the relations between quantity, numeration and place value, and any symbolic and concrete representations of the task.

CHILDREN'S ADDITION STRATEGIES

There is a large body of research which is concerned with describing and understanding young children's solutions to simple addition problems (cf. Carpenter, 1985; DeCorte & Verschaffel, 1987; Siegler, 1987). De Corte and Verschaffel (1987) compared their results, for strategies used by children in first grade to solve subtraction and addition problems, with those of Carpenter & Moser (1984). Carpenter and Moser's scheme for classifying children's solution strategies had two dimensions; first strategies were identified as additive or subtractive and then ordered according to the level of internalization; material (using objects), verbal (using counting), and mental (using known number facts). De Corte and Verschaffel identified similar levels of internalization.

There appears to be a remarkable amount of consistency in the literature regarding the strategies that young children use to solve addition problems. We classified the strategies according to whether they were material, verbal or mental (cf. De Corte & Verschaffel, 1987). After examination of our data we collapsed some categories, added others and used the codes for data analysis. In summary the strategies were as follows MATERIAL; CAA (counting all with analogs), CAJ (counting all in two sets with or without joining), COA (counting on from first or larger number), VERBAL; CA (counting all without analogs, COWA (counting on without analogs starting from the first or larger), MENTAL; RF (Recall of memorised facts), DF (Use of known or derived facts), OTHER; WAA (Written algorithm with analogs), WAWA (Written algorithms without analogs), PVA (Place value with analogs), PVWA (Place value without analogs), IM (Inappropriate methods), UNK (Indeterminate methods).

METHOD

Sample This consisted of 55 children, 18 in year 1, 19 in year 2, and 18 in year 3 in three suburban schools in Brisbane in low, medium and high socioeconomic areas.

Design Each teacher was asked to identify in advance those aspects of addition that she intended to teach and the representations and strategies that she intended to use. Two interviews with each child were conducted in term 2 and term 4 and separated by 10 weeks approximately to determine knowledge of content, and use of representations and strategies. All interviews with the children were videotaped and subsequently viewed, transcribed by the interviewer and then categorized and coded in consultation with the authors.

Test Items The test items were as follows; **1**. 3+1; **2**. 5+1; **3**. 2+0; **4**. 6+0; **5**. 4+3; **6**. 2+5; 7. 3+3; **8**. 4+4; **9**. 12+3; **10**. 21+4; **11**. 2+10; **12**. 34+5; **13**. 14+10; **14**. 9+7; **15**. 6+8; **16**. 12+21; **17**. 13+31; **18**. 17+5; **19**. 34+6; **20**. 121+324; **21**. 241+312; **22**. 265+126; **23**. 173+119; **24**. 143+281; **25**. 423+294. **26**. 166+254; **27**. 124+387. These required addition of 1, 0, doubles and other combinations within 10, addition of one digit to two digits with and without regrouping, and two and three digits with and without regrouping.

Materials These were selected on the basis of what the teachers said they intended to use, and included Multibase Arithmetic Blocks (MAB in base 10), sticks singly and in bundles of 10, counters, Unifix, the symbols " = " and " + " and the numerals for all problems written on cards, and paper and pencil for written calculations.

Procedure The child and the interviewer discussed the materials and their use. The child was shown the first appropriate addition task in horizontal form and asked to read it. If the child could not read the numerals they were read and discussed by the interviewer. It was suggested that he or she could use fingers, materials or paper and pencil to represent the operation and find the answer. The child was encouraged to talk about the process. Testing began at an appropriate item for each year level. Each child stopped after difficulty with three tasks in succession.

RESULTS

Teacher expectations and strategies Teachers said they followed the recommendations in the State curriculum. These will be summarized.

Children's responses A MANOVA for time by year was computed for the categories CAA, CAJ, CA, COA, COWA, RF, DF, WAA, WAWA, PVA, PVWA, IM and UNK. The overall year by time effect was significant (Pillai's trace =.94, F=2.5, DF=2,52, p=.001). The univariate analyses for year by time showed that differences for CAJ (counting all in two sets with or without joining), RF (recall of memorised facts), DF (use of known and derived facts) and PVWA (place value without analogs) were significant, F=4.9 (p=.01), F=10.3 (p=.000), F=5.9 (p=.005) and F=4.7 (p=.005) respectively. Tukey post hoc analyses were used to confirm honestly significant differences between groups early and late in the year in these categories. These differences are shown developmentally in Figures 1A, 1B, 1C and 1D. These will be discussed.

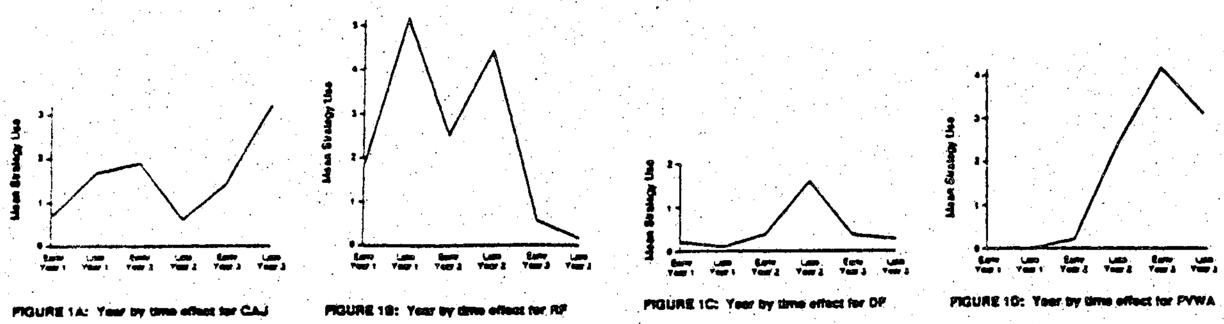

FIGURE 1 Significant differences for year by time

The univariate effects by year were significant for CAA (counting all with analogs), COA (counting on from the first or larger number), COWA (counting on without analogs), RF (recall of memorised facts), DF (use of known or derived facts), PVWA (place value without analogs), and IM (inappropriate methods). Tukey post hoc analyses indicated honestly significant differences at the 0.05 level for CAA (Fig 2A), for COA (Fig 2B), for COWA (Fig 2C), for RF (Fig 2D), for DF (Fig 2E), for PVWA (Fig 2F), and for IM (Fig 2G). There were no significant differences between years for use of written algorithms with or without analogs although there was an increase in their use in year 3. The developmental trends shown in Figures 2A to 2G show that children generally chose to use material strategies and count until year 2, to use the verbal strategies of recall of facts and derived facts until year 2, and then to use place value explanations (mental) or joining sets of objects and counting (material) as frequently at least as written algorithms with or without analogs in year 3 despite the fact that had been `taught' to use them.

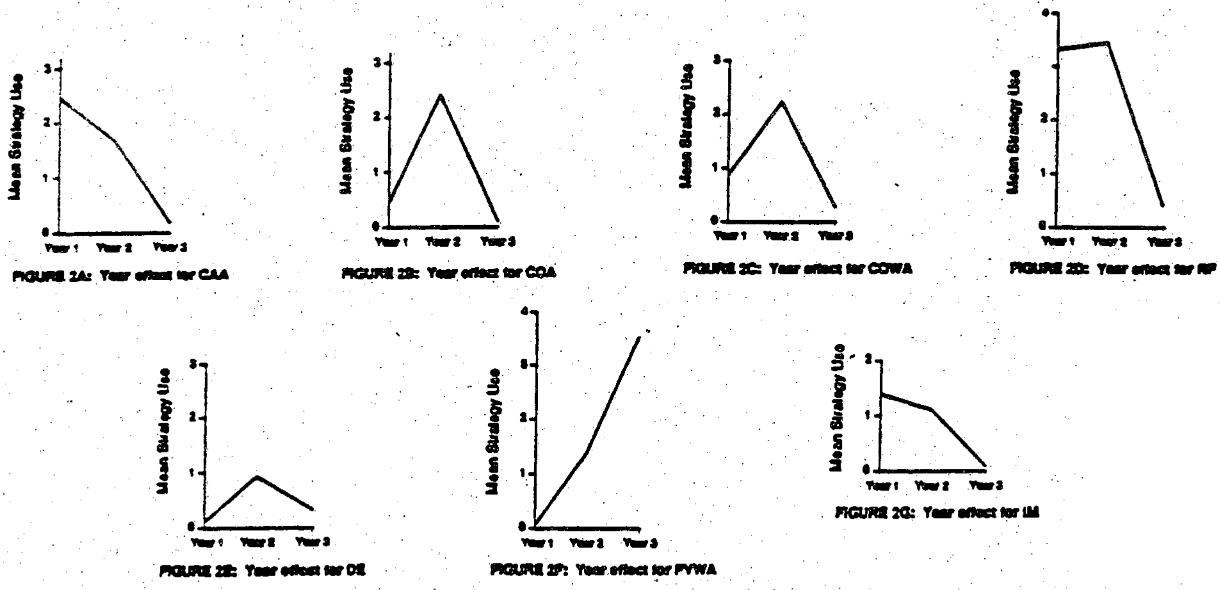

FIGURE 2 Significant differences by year

An analysis of strategy use by item confirmed the developmental trends obtained from the MANOVA. It also showed very clearly that children change strategy use to suit their confidence and perception of the difficulty of the task as proposed by (Siegler & Robinson, 1982; Russell, 1977).

The items on which children were incorrect 50% or more of the time were as follows 2+0, 12+21, 13+31, 121+324, 265+126, 173+119, 281+143, 387+124 . These errors and time of occurence will be discussed.

Most frequently children chose to use no analogs at all. The second most frequently chosen analog was fingers. There was limited use of MAB in years 2 and 3.

The oneway analysis of variance of categories by school showed no significant differences between schools.

DISCUSSION

The results indicate that children preferred to use verbal and mental strategies rather than formal algorithms and did not want to use analogs unless they could not perform the task in any other way. The results are encouraging in that children are making sense of the operations using their own strategies and apparently reducing the processing load by using no materials or materials only. They are disheartening if the objective is to have children relate their previous knowledge of addition to formal written procedures. Reasons for these results will be suggested and implications for teaching discussed.

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